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The limitset of a Coxeter group and a Cannon-Thurston map

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1 Introduction

A new dynamical approach to analyze the asymptotic behavior of the root system associating a Coxeter group has been introduced by Hohlweg, Labbé and Ripoll in [10]. This approach implicate a study of infinite Coxeter groups from a dynamical viewpoint. For the case where the associated matrices have signature $(n-1,1)$, Coxeter groups also act on hyperbolic space in the sense of Gromov.

Let (X,d_X) and (Y,d_Y) be metric spaces equipped with an action of a countable group G respectively. A map $f:X\longrightarrow Y$ is called G-equivariant if f satisfies

$$g\circ f(x)=f\circ g(x)$$

holds for all $x\in X$ and for all $g\in G$.

In general, a continuous equivariant between boundaries of a discrete group and their limit set is called a Cannon-Thurston map. In this article we shall consider whether the Cannon-Thurston map for the Coxeter groups exists.

Theorem 1.1. *Let W be a rank n Coxeter groups whose associating bi-linear form B has the signature $(n-1,1)$. Let $\partial_G W$ be the Gromov boundary of W and let $\Lambda(W)$ be the limit set of W. There exists a W-equivariant, continuous surjection $F:\partial_G W\longrightarrow\Lambda(W)$.*

We remark that the Gromov boundary is ordinary defined on a hyperbolic metric space. We extend the definition to arbitrary metric space by taking transitive closure due to Buckley and Kokkendorff ([3]). The limit set of a Coxeter subgroup W' generated by a subset S' of S are located on ∂D. In fact the set of basis Δ' corresponding to W' is a subset of Δ and the limit set of W' is distributed on convex hull of Δ'. This fact leads the following corollary:

Corollary 1.2. *Let (W,S) be a Coxeter system of rank n whose associated bi-linear form has the signature $(n-1,1)$. For a special subgroup W' whose associated bi-linear form has the signature $(n-1,1)$, if the normalized action (see §2) of W' is cocompact, then the limit set $\Lambda(W')$ of W' is canonically embedded into the limit set of $\Lambda(W)$.*

2 The Coxeter systems and geometric representation

2.1 The Coxeter systems

A *Coxeter group* W of rank n is generated by the set $S = \{s_1, \dots, s_n\}$ with the relations $(s_i s_j)^{m_{ij}} = 1$, where $m_{ij} \in \mathbb{Z}_{>1} \cup \{\infty\}$ for $1 \leq i < j \leq n$ and $m_{ii} = 1$ for $1 \leq i \leq n$. More precisely, we say that the pair (W, S) is a *Coxeter system.*

For a Coxeter system (W, S) of rank n, let V be a real vector space with its orthonormal basis $\Delta = \{\alpha_s | s \in S\}$ with respect to the Euclidean inner product. Note that by identifying V with $\mathbb{R}^n$, we treat V as a Euclidean space. We define a symmetric bilinear form on V by setting

$$B(\alpha_i, \alpha_j) \begin{cases} = -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } m_{ij} < \infty, \\ \leq -1 & \text{if } m_{ij} = \infty \end{cases}$$

for $1 \leq i \leq j \leq n$, where $\alpha_{s_i} = \alpha_i$, and call the associated matrix B the *Coxeter matrix.* Classically, $B(\alpha_i, \alpha_j) = -1$ if $m_{ij} = \infty$, but throughout this thesis, we allow its value to be any real number less than or equal to -1. This definition derives from [10]. Given $\alpha \in V$ such that $B(\alpha, \alpha) \neq 0$, s_α denotes the map $s_\alpha : V \to V$ by

$$s_\alpha(v) = v - 2\frac{B(\alpha, v)}{B(\alpha, \alpha)}\alpha \quad \text{for any } v \in V,$$

which is said to be a *B-reflection.* Then Δ is called a *simple system* and its elements are *simple roots* of W. The Coxeter group W acts on V associated with its generating set S as compositions of B-reflections $\{s_\alpha \mid \alpha \in \Delta\}$ generated by simple roots. The *root system* Φ of W is defined to be the orbit of Δ under the action of W and its elements are called its *roots.* Let

$$V^+ := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^n v_i \alpha_i, v_i > 0 \right\}, \quad V^- := \left\{ v \in V \;\middle|\; v = \sum_{i=1}^n v_i \alpha_i, v_i < 0 \right\}.$$

Assumption 2.1. In this paper, we always assume the following.

- The bilinear form B has the signature $(n-1, 1)$. We call such a group a Coxeter group of type $(n-1, 1)$.
- The Coxeter matrix$B$ is not block-diagonal up to permutation of the basis. In that case, the matrix B is said to be *irreducible.*

It turns out that we only need to work on the case where B is irreducible. If the matrix B is reducible, then we can divide Δ into l subsets $\Delta = \sqcup_{i=1}^l \Delta_i$ so that each corresponding matrix $B_i = \{B(\alpha, \beta)\}_{\alpha, \beta \in \Delta_i}$ is irreducible and B is block diagonal $B = (B_1, \dots, B_l)$. Then for any distinct i, j, if $\alpha \in \Delta_i$ and $\beta \in \Delta_j$, s_α and s_β commute. In this case we see that W is direct product

$$W = W_1 \times W_2 \times \cdots \times W_l,$$

where W_i is the Coxeter group corresponding to Δ_i. From this, the action of W can be regarded as a direct product of the actions of each W_i. Moreover if B has the signature $(n-1,1)$, there exists a unique B_k which has the signature $(n_k-1,1)$ and others are positive definite. Since if the Coxeter matricis positive definite then the corresponding Coxeter group W' is finite, and hence the limit set $\Lambda(W')=\emptyset$ (for the definition of the limit set, see Section 3.3). This ensures that $\Lambda(W)$ is distributed on $\mathrm{conv}(\widehat{\Delta_k})$, where $\mathrm{conv}(\widehat{\Delta_k})$ is the convex hull of $\widehat{\Delta_k}$. Thus $\Lambda(W)=\Lambda(W_k)$. Accordingly, if there exists the Cannon-Thurston map for W_k then we also have the Cannon-Thurston map for the whole group W. This follows from the fact that the direct product $G_1\times G_2$ of a finite generated infinite group G_1 and a finite group G_2 has the same Gromov boundary as that of G_1.

Lemma 2.2. *Let o be an eigenvector for the negative eigenvalue of B. Then all coordinates of o have the same sign.*

This follows from Perron-Frobenius theorem for irreducible non-negative matrices. In fact, letting I be the identity matrix of rank n, we apply Perron-Frobenius theorem to an $-B+I$ irreducible and non-negative. Then the result easily follows.

We fix $o\in V$ to be the eigenvector corresponding to the negative eigenvalue of B whose euclidean norm equals to 1 and all coordinates are positive. Hence if we write o in a linear combination $o=\sum_{i=1}^{n}o_i\alpha_i$ of Δ then $o_i>0$. Given $v\in V$, we define $|v|_1$ by $\sum_{i=1}^{n}o_iv_i$ if $v=\sum_{i=1}^{n}v_i\alpha_i$. Note that a function $|\cdot|_1: V\to\mathbb{R}$ is actually a norm in the set of vectors having nonnegative coefficients. It is obvious that $|v|_1>0$ for $v\in V^+$ and $|v|_1<0$ for $v\in V^-$. Let $V_i=\{v\in V\mid |v|_1=i\}$, where $i=0,1$. For $v\in V\setminus V_0$, we write $\widehat{v}$ for the "normalized" vector $\frac{v}{|v|_1}\in V_1$. We also call o the normalized eigenvector (corresponding to the negative eigenvalue of B). Also for a set $A\subset V\setminus V_0$, we write $\widehat{A}$ for the set of all $\widehat{a}$ with $a\in A$. We notice that $B(x,\alpha)=|\alpha|_1B(x,\widehat{\alpha})$ hence the sign of $B(x,\alpha)$ equals to the sign of $B(x,\alpha)$ for any $x\in V$ and $\alpha\in\Delta$.

We denote $q(v)=B(v,v)$ for $v\in V$. Let $Q=\{v\in V\mid q(v)=0\}$, $Q_-=\{v\in V\mid q(v)<0\}$ then we have

$$\widehat{Q}=V_1\cap Q,\quad \widehat{Q_-}=V_1\cap Q_-.$$

Since B is of type $(n-1,1)$, $\widehat{Q}$ is an ellipsoid. The cone Q_- has two components the "positive side" Q_-^+, that is the component including o, and the "negative side" $Q_-^-=-Q_-^+$. Similarly we divide Q into two components Q^+ and Q^- so that $Q^+=\partial Q_-^+$ and $Q^-=\partial Q_-^-$.

Remark 2.3. We have

$$W(V_0)\cap Q=\{\mathbf{0}\},$$

where $\mathbf{0}$ is the origin of $\mathbb{R}^n$. To see this we only need to verify that $V_0\cap Q=\{\mathbf{0}\}$ since Q is invariant under B-reflections. We notice that $V_0=\{v\in V\mid B(v,o)=0\}$. For $i=1,\ldots,n-1$, let p_i be an eigenvector of B corresponding to a positive eigenvalue λ_i. For any $v \in V_0$, we can express v in a linear combination $v = \sum_i^{n-1} v_i p_i$ since $B(v, o) = 0$. Then we have $B(v, v) = \sum_i^{n-1} \lambda_i v_i^2 \|p_i\|^2 \geq 0$ where $\| * \|$ denotes the euclidean norm. Since $\lambda_i > 0$ for $i = 1, \ldots, n-1$, we have $B(v, v) = 0$ if and only if $v = \mathbf{0}$.

2.2 The word metric

Let G be a finitely generated group. Fixing a finite generating set S of G, all elements in G can be represented by a product of elements in $S \cup S^{-1}$ where $S^{-1} = \{s^{-1} \mid s \in S\}$. We say such a representation to be a *word.* Letting $\langle S \rangle$ be the set of words. For a word $w \in \langle S \rangle$ we define the *word length* $\ell_S(w)$ as the number of generators $s \in S$ in w. Now, we naturally have a map $\iota : \langle S \rangle \longrightarrow W$. For a given $g \in G$, we define the *minimal word length* $|g|_S$ of g by $\min\{\ell_S(w) \mid w \in \iota^{-1}(g)\}$. An expression of g realizing $|g|_S$ is called *the reduced expression* or *the geodesic word.* Using the word length, we can define so-called *the word metric* with respect to S on G, i.e. for $g, h \in G$, their distance is $|g^{-1}h|_S$.

3 The Hilbert metric

3.1 The cross ratio and the Hilbert metric

For four vectors $a, b, c, d \in V$ with $c - d, b - a \neq 0$, we define *the cross ratio* $[a, b, c, d]$ with respect to B by

$$[a, b, c, d] := \frac{\|y - a\| \, \|x - b\|}{\|y - b\| \, \|x - a\|},$$

where $\| * \|$ denotes the Euclidean norm. Using this we obtain a distance d on D as follows. For any $x, y \in D$, take $a, b \in \partial D$ so that the points a, x, y, b lie on the segment connecting a, b in this order. Then $y - b, x - a \neq 0$. We define a function d as follows.

$$d(x, y) := \log[a, x, y, b],$$

This is actually a metric on D and called *the Hilbert metric* on D.

3.2 Some properties of the Hilbert metric

In this section we correct known geometric properties of a space with the Hilbert metric.

Proposition 3.1. *(D, d_D) is*

(i) a proper (i.e. any closed ball is compact) complete metric space and,

(ii) a uniquely geodesic space.

Let (X, d) be a geodesic space. For $x, y, p \in X$, we define the *Gromov product* $(x|y)_p$ of x and y with respect to p by the equality

$$(x|y)_p = \frac{1}{2}\left(d(x,p) + d(y,p) - d(x,y)\right).$$

Using this, the hyperbolicity in the sense of Gromov is defined as follows. For $\delta \geq 0$ the space X is *δ-hyperbolic* if

$$(x|z)_p \geq \min\{(x|y)_p, (y|z)_p\} - \delta$$

for all $x, y, z, p \in X$. We say the space is simply *Gromov hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$.

A metric space (D, d_D) with the Hilbert metric is a CAT(0) and Gromov hyperbolic space since the region D is an ellipsoid. The former derived from a result given in [6] by Egloff.

Theorem 3.2 (Egloff). *Let $H \subset \mathbb{R}^n$ be a convex open set with the Hilbert metric d_H. Then (H, d_H) is a CAT(0) space if and only if H is an ellipsoid.*

The latter owe to a result of Karlsson-Noskov [?].

Theorem 3.3 (Karlsson-Noskov). *Let $H \subset \mathbb{R}^n$ be a convex open set with the Hilbert metric d_H. If H is an ellipsoid, then (H, d_H) is a Gromov hyperbolic.*

The point of our definition of the Hilbert metric can be seen in the proof of the following proposition.

Proposition 3.4. *Let W be a Coxeter group with signature $(n-1, 1)$. The normalized action of any $w \in W$ is an isometry on (D, d_D).*

4 The properness of the normalized action

We verify that the normalized action on (D, d_D) is proper. If X is locally compact and there exists a fundamental region R then the action is proper.

We define two open sets (with respect to the subspace topology of V_1)

$$K := \{v \in D \mid \forall \alpha \in \Delta, B(\alpha, v) < 0\} \qquad \text{and} \qquad K' := K \cap D'.$$

For $\alpha \in \Delta$ we set $P_\alpha = \{v \in V_1 \mid \alpha\text{-th coordinate of } v \text{ is } 0\}$ and $H_\alpha = \{v \in V_1 \mid B(v, \alpha) = 0\}$. We define

$$\mathcal{P} = \{v \in V_1 \mid \forall \alpha \in \Delta, B(\alpha, v) < 0\} \qquad \text{and} \qquad \mathcal{P}' = \mathcal{P} \cap \operatorname{int}(\operatorname{conv}(\widehat{\Delta})).$$

Then clearly $K = \mathcal{P} \cap D$. Moreover, we will see that $K' = \mathcal{P}' \cap D$ (Lemma ??). Since $\mathcal{P}$ (resp. $\mathcal{P}'$) is bounded by finitely many $n-1$ dimensional subspaces $\{H_\alpha \mid \alpha \in \Delta\}$ (resp. $\{H_\alpha \mid \alpha \in \Delta\}$ and $\{P_\alpha \mid \alpha \in \Delta\}$), actually $\overline{\mathcal{P}}$ (resp. $\overline{\mathcal{P}'}$) is a polyhedron. In general, $\mathcal{P}$ is not a simplex. The following example of W such that $\mathcal{P}$ is not a simplex is given by Yohei Komori.

$$W = \langle s_1, \ldots, s_5 \mid s_i^2, (s_{i-1}s_i)^4 \rangle,$$

where $i = 1, \ldots, 5$ and $s_0 = s_5$.

Definition 4.1. We assume that a group G acts on a metric space X isometrically. We denote the action by $g.x$ for $g \in G$ and $x \in X$. Then an open set $A \subset X$ is a *fundamental region* if $\overline{G.A} = X$ and $g.A \cap A = \emptyset$ for any $g \in G$ where $\overline{G.A}$ is the topological closure of $G.A$.

Proposition 4.2. *K is a fundamental region for the normalized action.*

Definition 4.3. Let (W, S) be a Coxeter system.

- We call a sequence $\{w_k\}_k$ in W a *short sequence* if for each $n \in \mathbb{N}$ there exists $s \in S$ such that $w_{k+1} = sw_k$ and $|w_k| = k$.
- For a sequence $\{w_k\}_k$ in W, a path in V_1 is a *sequence path* for $\{w_k\}_k$ if the path is given by connecting Euclidean segments $[w_k \cdot o, w_{k+1} \cdot o]$ for all $k \in \mathbb{N}$.

The following is a key of our argument.

Proposition 4.4. *Suppose that W acts on D cocompactly. For any $\xi \in \Lambda(W)$ there exists a short sequence $\{w_k\}_k$ so that $w_k \cdot o$ converges to ξ. Furthermore the sequence path for $\{w_k\}_k$ lies in c-neighborhood of a segment $[o, \xi]$ connecting o and ξ for some $c > 0$ with respect to the Hilbert metric.*

4.1 Three cases

We consider the normalized action by dividing it into the following three cases; cocompact, convex cocompact, with cusps. We recall that $\operatorname{conv}(\widehat{\Delta})$ is a simplex. It can happen three distinct situations due to the bilinear form B;

(i) the region $D \cup \partial D$ is included in $\operatorname{int}(\operatorname{conv}(\widehat{\Delta}))$;

(ii) there exist some n' ($< n$) dimensional faces of $\operatorname{conv}(\widehat{\Delta})$ which are tangent to the boundary ∂D;

(iii) $D \cup \partial D \not\subset \operatorname{int}(\operatorname{conv}(\widehat{\Delta}))$ and no faces of $\operatorname{conv}(\widehat{\Delta})$ tangent to ∂D.

We argue the cases (i) and (iii) simultaneously. For the case (ii), we can not apply the same argument as (i) and (iii). The most general case will be discussed in Section 4.2.

Remark 4.5. By [8, Corollary 2.2], we see that a Coxeter subsystem (W', S') satisfying $S' \subset S$ is either of type $(|S'| - 1, 1)$ or $(|S'| - 1, 0)$ or positive definite. Let B' be the bilinear form corresponding to (W', S'). If B' has the signature $(|S'| - 1, 1)$ (resp. $(|S'| - 1, 0)$), then by the same argument as Lemma 2.2, we have an eigenvector $o' \in \operatorname{span}(\Delta')$ of the negative eigenvector (resp. 0 eigenvalue) such that all coordinates of o' for Δ' are positive where $\operatorname{span}(\Delta')$ denotes the subspace spanned by Δ'. This shows that $Q' = \{v \in \operatorname{span}(\Delta') \mid B'(v, v) = 0\}$ should intersect with $\operatorname{conv}(\widehat{\Delta'})$. Since the Coxeter matrix of B' is a principal submatrix of the Coxeter matrixof B, we see that $\partial D \cap \operatorname{conv}(\widehat{\Delta'}) = Q' \cap \operatorname{conv}(\widehat{\Delta'})$. Thus we have the followings:

(1) B' has the signature $(|S'|-1,1)$ if and only if $D \cap \mathrm{conv}(\Delta') \neq \emptyset$;

(2) B' has the signature $(|S'|-1,0)$ if and only if $\partial D \cap \mathrm{conv}(\Delta') = Q' \cap \mathrm{conv}(\widehat{\Delta'})$, which is a singleton;

(3) B' is positive definite if and only if $(D \cup \partial D) \cap \mathrm{conv}(\widehat{\Delta'}) = \emptyset$.

If B' has the signature $(|S'|-1,1)$ then H_α for $\alpha \in \Delta'$ intersects with $D \cap \mathrm{conv}(\Delta')$. In fact if not, then $D \cap \mathrm{conv}(\widehat{\Delta'})$ is not preserved by s_α for $\alpha \in \Delta'$. Moreover, by the compactness of Q, $Q' \cap V_0 = \mathbf{0}$ for any Coxeter subsystem (W', S').

We say a Coxeter system of rank n is *affine* if its associating bi-linear form B has the signature $(n-1,0)$. Fixing a generating set S we simply say Coxeter group W is affine if the Coxeter system (W,S) is affine. An affine Coxeter group is of infinite order and its limit set is a singleton ([10, Corollary 2.15]).

By a simple argument using the linearity of the original action of Coxeter groups, we can rephrase these cases as follows.

Proposition 4.6. *For each case, we have the followings:*

(*a*) *The case* (*i*)	$\Longleftrightarrow$	$\overline{K'} = \overline{K} \subset D$,
	$\Longleftrightarrow$	*every Coxeter subgroup of W of rank $n-1$ generated by a subset of S is finite:*
(*b*) *The case* (*ii*)	$\Longleftrightarrow$	$\overline{K}$ *or* $\overline{K'}$ *has some vertices in* ∂D,
	$\Longleftrightarrow$	*W includes at least one affine special subgroup:*
(*c*) *The case* (*iii*)	$\Longleftrightarrow$	*all the vertices of $\overline{K}$ are not always in ∂D and at least one of them is not in D,*
	$\Longleftrightarrow$	*every special subgroup of W of rank n' $(< n)$ is of type $(n'-1,1)$ or $(n',0)$.*

From Proposition 4.6 we deduce that the fundamental region K (resp.K') is bounded if the case (i) (resp. the case (ii)) occurs. If $\overline{K'}$ is not compact, then ∂D must be tangent to some faces of $\mathrm{conv}(\widehat{\Delta})$. In this case K' has some cusps at points of tangency of ∂D. This happens if and only if (ii). Because of this we call each cases as follows: The normalized action of W on D is

- *cocompact* if the case (i) happens;
- *with cusps* if the case (ii) happens;
- *convex cocompact* if the case (iii) happens.

In the case (ii) the *rank* of cusp v is the minimal rank of the affine Coxeter subgroup generated by a subset of S which fixes v.

Note that we can find easily that there exist Coxeter groups corresponding to each cases (i), (ii) and (iii). Thus all the possibilities may happen.

Example 4.7. We see that classical hyperbolic Coxeter groups are in the case (i). For the case (iii) one of the simplest example is a triangle group $W =$

$\langle s_1, s_2, s_3 \mid s_i^2\ (i = 1,2,3)\rangle$ with bi-linear form satisfying $B(\alpha_i, \alpha_j) < -1$ for $i \neq j$. At last it is in the case (ii) that $W = \langle s_1, s_2, s_3, s_4 \mid s_i^2, (s_1s_2)^6, (s_1s_3)^3, (s_js_k)^2 (j \neq k \in \{2,3,4\})\rangle$ with the matrix $(B(\alpha_i, \alpha_j))_{i,j}$ equals to

$$\begin{bmatrix} 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & T \\ -\frac{\sqrt{3}}{2} & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & 1 & 0 \\ T & 0 & 0 & 1 \end{bmatrix}$$

where $T < -1$. In fact W is with signature $(3,1)$ although a subgroup generated by $\{s_1, s_2, s_3\}$ is with signature $(2,0)$.

Definition 4.8 (The limit set). For a Coxeter system (W,S) of type $(n-1,1)$, let o be the normalized eigenvector corresponding to the negative eigenvalue of the corresponding Coxeter matrix. The *limitset* $\Lambda_B(W)$ of W with respect to B is the set of accumulation points of the orbit of o by the normalized action of W on D in the Euclidean topology. The limit set depends on the Coxeter matrix$B$. If B is understood, then we simply denote the limit set by $\Lambda(W)$.

5 Two boundaries of spaces

5.1 The Gromov boundaries

The Gromov boundary of a hyperbolic space is one of the most studied boundary at infinity. In this section we define it for an arbitrary metric space due to [3].

Let (X,d,o) be a metric space with a base point o. We denote simply $(*|*)$ as the Gromov product with respect to the base point o. A sequence $x = \{x_i\}_i$ in X is a *Gromov sequece* if $(x_i|x_j)_z \longrightarrow \infty$ as $i,j \longrightarrow \infty$ for any base point $z \in X$. Note that if $(x_i|x_j)_z \longrightarrow \infty$ $(i,j \longrightarrow \infty)$ for some $z \in X$ then for any $z' \in X$ we have $(x_i|x_j)_{z'} \longrightarrow \infty$ $(i,j \longrightarrow \infty)$.

We define a binary relation $\sim_G$ on the set of Gromov sequences as follows. For two Gromov sequences $x = \{x_i\}_i, y = \{y_i\}_i$, $x \sim_G y$ if $\liminf_{i,j\longrightarrow\infty}(x_i|y_j) = \infty$. Then we say that two Gromov sequences x and y are equivalent $x \sim y$ if there exist a finite sequence $\{x = x_0, \ldots, x_k = y\}$ such that

$$x_{i-1} \sim_G x_i \quad \text{for} \quad i = 1, \ldots, k.$$

It is easy to see that the relation $\sim$ is an equivalence relation on the set of Gromov sequences. The *Gromov boundary* $\partial_G X$ is the set of all equivalence classes $[x]$ of Gromov sequences x. If the space X is a finitely generated group G then the Gromov boundary of G depends on the choice of the generating set in general. In this thesis we always define the Gromov boundary of a Coxeter group W using the generating set of the Coxeter system (W,S). We shall use without comment the fact that every Gromov sequence is equivalent to each of its subsequences. To simplify the statement of the following definition, we denote a point $x \in X$ by the singleton equivalence class $[x] = [\{x_i\}_i]$ where $x_i = x$ for all

i. We extend the Gromov product with base point o to $(X \cup \partial_G X) \times (X \cup \partial_G X)$ via the equations

$$(a|b) = \begin{cases} \inf\{\liminf_{i,j \longrightarrow \infty}(x_i|y_j) \mid [x] = a, \ [y] = b\}, & \text{if } a \neq b, \\ \infty, \quad \text{if } a = b. \end{cases}$$

We set

$$U(x,r) := \{y \in \partial_G X \mid (x|y) > r\}$$

for $x \in \partial_G X$ and $r > 0$ and define $\mathcal{U} = \{U(x,r) \mid x \in \partial_G X, r > 0\}$. The Gromov boundary $\partial_G X$ can be regarded as a topological space with a subbasis $\mathcal{U}$.

If the space X is δ-hyperbolic in the sense of Gromov, then this topology is equivalent to a topology defined by the following metric. For $\epsilon > 0$ satisfying $\epsilon\delta \leq 1/5$, we define d_ϵ as follows:

$$d_\epsilon(a,b) = e^{-\epsilon(a|b)} \quad (a, b \in \partial_G X).$$

Then it is known that d_ϵ is actually a metric. In this thesis, we always take ϵ so that $\epsilon\delta \leq 1/5$ for all δ hyperbolic spaces X and assume that $\partial_G X$ is equipped with d_ϵ-topology.

5.2 The CAT(0) boundaries

The map we want is given via the *CAT(0) boundary* $\partial_I D$ (or $\partial_I D'$) with the cone topology of D (or D'). That is a space of geodesic rays emanating from a base point. Consult with [2] for the precise definition.

Since the region D' and D are both complete CAT(0) space, CAT(0) boundaries for each space are well defined. We use the eigenvector o for the negative eigenvalue as the base point in the definition of CAT(0) boundary and the cone topology. Furthermore since D' is a subspace of D, its CAT(0) boundary $\partial_I(D')$ is a subspace of $\partial_I D$.

$\partial_I D$ (resp. $\partial_I D'$) is homeomorphic to ∂D (resp. $\partial D' \setminus D$).

Remark 5.1. If the case space X is a complete proper hyperbolic CAT(0) space then $\partial_G X \simeq \partial_I X$ ([3, Theorem 2.2 (d)]). Because of this, if the case (i) (resp. the case (iii)) happens then $\partial_I D \simeq \partial_G D$ (resp. $\partial_I D' \simeq \partial_G D'$).

Remark 5.2. If the case (iii) happens, then $\Lambda(W)$ is homeomorphic to $\partial D' \setminus D$. Moreover we see that $\Lambda(W) = \partial D' \setminus D \simeq \partial_I D' \simeq \partial_G D'$.

6 The Cannon-Thurston maps

In this section, we give a proof of Theorem 1.1. Throughout this section, a vector o denotes the normalized (with respect to $| * |_1$) eigenvector corresponding to the negative eigenvalue of B.

6.1 The case of W acting without cusps

We consider when W acts cocompactly or convex cocompactly. In this case W is hyperbolic in the sense of Gromov. For simplicity, we mean $\widetilde{D}$ for D or D'. Our purpose in this section is actually to construct a homeomorphism from $\partial_G(W, S)$ to $\partial\widetilde{D}$. We define the map $f : W \longrightarrow \widetilde{D}$ by $w \mapsto w \cdot o$ where o is the eigenvector of the negative eigenvalue. This map is a quasi-isometry.

It is well known that f extends to a homeomorphism between $\partial_G(W, S) \cup W$ and $\partial_G\widetilde{D} \cup \widetilde{D}$. Let $\overline{f}$ be the restriction of the homeomorphism above to $\partial_G W$. Now we recall following two maps. By the result of Buckley and Kokkendorff [3], we know that there exists a homeomorphism $g : \partial_G\widetilde{D} \longrightarrow \partial_I\widetilde{D}$. Moreover, for a Gromov sequence $\xi \in \partial_G\widetilde{D}$ any unbounded sequence given as a subset of a geodesic ray $g(\xi)$ is equivalent to ξ. On the other hand we have a homeomorphism $h : \partial_I\widetilde{D} \longrightarrow \partial\widetilde{D}$.

We compose these homeomorphisms. Let $F = h \circ g \circ \overline{f}$. Then we have a homeomorphism from $\partial_G(W, S)$ to $\partial\widetilde{D}$. We verify that F sends $\omega \in \partial_G(W, S)$ to the limit point defined by $\{w_k \cdot o\}_k$ for $\{w_k\}_k \in \omega$. If this is true, then we see that F is W-equivariant by the construction. To see this, we inspect the details of the maps g and h. For our situation, the proof in [3] says that for a Gromov sequence $\{w_k \cdot o\}_k \in F([\{w_k\}_k])$ in W, there exists a ξ such that a sequence $\{u_i \cdot o\}_i$ constructed by the same way as in the proof of Proposition 4.4 is a short sequence included in a bounded neighborhood of ξ. The image of ξ by h is equivalent to $\{u_i \cdot o\}_i$ in the sense of Gromov. Adding to this, Buckley and Kokkendorff showed that $\{u_i \cdot o\}_i$ equivalent to the original sequence $\{w_k \cdot o\}_k$ and hence they converge to the same point in $\partial_G\widetilde{D} \setminus D$. By Remark 5.2 F is the map we want.

6.2 The case of W acting with cusps

We know that there exist some Coxeter groups acting on D with cusps. By Proposition 4.6, this happens when ∂D is tangent to some faces of $\mathrm{conv}(\Delta)$. We divide this case into following three cases;

(i) there exists at least one pair of simple roots $\alpha, \beta \in \Delta$ so that $B(\alpha, \beta) = -1$,

(ii) there exists at least one subset $\Delta' \subset \Delta$ whose cardinality is more than 3 so that the corresponding matrix B' is positive semidefinite (not positive definite) where B' is the matrix obtained by restricting B to Δ',

(iii) or (i) and (ii) happen simultaneously.

The case (i).
We deal with the case (i) first. In this case, the dihedral subgroup of W generated by s_α and s_β is infinite and its limit set is one point. This means that D is tangent to the segment connecting α and β. Hence the fundamental region of W is unbounded.

For the cases (ii) and (iii), we have to see other geometric aspects of the Coxeter groups.

Recall that the number n is the rank of W and hence equals to the dimension of V. Let $\{A_m\}_m$ be a sequence of $n \times n$ matrices which are defined as follows. For each $m \in \mathbb{N}$, we define A_m so that

$$A_m(\alpha, \beta) = \begin{cases} 1/m, & \text{if } B(\alpha, \beta) = -1, \\ 0, & \text{if otherwise,} \end{cases}$$

for each $\alpha, \beta \in \Delta$. We denote the bilinear form with respect to each A_m by $A_m(v, v')$ for $v, v' \in V$. Then let $B_m = B - A_m$.

If B has the signature $(n-1, 1)$, then B_m also has the signature $(n-1, 1)$ for sufficiently large $m \in \mathbb{N}$. Therefore for sufficiently large m, our definitions of Q,D, D', L, K can be extended to the bilinear form defined by B_m. We define Q_m, D_m, D'_m, L_m, K_m each of them by using B_m instead of B in their definitions. Clearly B_m converges to B as m tends to ∞.

Let $v_1, \ldots, v_n$ be eigenvectors of B normalized with respect to the Euclidean norm so that the matrix $(v_1, \ldots, v_n)$ diagonalize B. Then since each $P_{m,i}(v_i)$ converges to v_i, the matrix diagonalizing B_m also converges to $(v_1, \ldots, v_n)$. This fact shows that the sequence $\{D_m\}_m$ converges to D.

We can consider the B_m-reflection of W on V with respect to B_m. We denote this action by ρ_m. For example, the B_m-reflection of $\alpha \in \Delta$ can be calculated as

$$\rho_m(s_\alpha)(x) = x - 2B_m(x, \alpha)\alpha, \quad (x \in V).$$

The normalized action with respect to B_m is defined in the same way as B. We denote this also by ρ_m. Furthermore if B_m has the signature $(n-1, 1)$, then all our lemmas and propositions can be proved by using the normalized eigenvector o_m corresponding to the negative eigenvalue of B_m instead of o. Therefore if the normalized action ρ_m is (convex) cocompact, then there exists a map F_m from the Gromov boundary $\partial_G(W, S)$ of W to the limit set $\Lambda_{B_m}(W)$ which is homeomorphic. In fact we have a W-equivariant homeomorphism $F_m : \partial_G(W, S) \longrightarrow \Lambda_{B_m}(W)$ for each m since the case (iii) happens. Note that for sufficiently large m, we have $V_0 \cap Q_m = \{\mathbf{0}\}$. Hence we can define the Hilbert metric on $V_1 \cap Q_{m-}$ where $Q_{m-} = \{v \in V \mid B_m(v, v) < 0\}$. Consider the correspondence between $x \in D_m$ and $y = \mathbb{R}x \cap V_1 \cap Q_{m-}$. Then we see that this is an isometry between D_m and $V_1 \cap Q_{m-}$ and W equivariant. Thus we can regard the normalized action ρ_m as an action of W on $V_1 \cap Q_{m-}$.

We remark that for any $\alpha \in \Delta$ and $m \in \mathbb{N}$, we have $B_m(o, \alpha) = B(o, \alpha) - A_m(o, \alpha) < 0$ since $B(o, \alpha) < 0$ and all coordinates of o are positive. Hence o is in K_m for any $m \in \mathbb{N}$.

Proposition 6.1. *Assume that the normalized action of W includes rank 2 cusps. There exists a continuous W-equivariant surjection $\iota : \Lambda(\rho_1(W)) \longrightarrow \Lambda(W)$.*

Considering the composition $F' = \iota \circ F_1$, we have the map which is surjective, continuous and W-equivariant.

If $B(\alpha, \beta) = -1$ for some $\alpha, \beta \in \Delta$ then the Coxeter subgroup W' generated by $\{s_\alpha, s_\beta\}$ is affine. Since an affine Coxeter group has only one limit point,

$\{(s_\alpha s_\beta)^k \cdot o\}_k$ and $\{(s_\beta s_\alpha)^k \cdot o\}_k$ converges to the same limit point. However in the Gromov boundary of (W, S), $\{(s_\alpha s_\beta)^k\}_k$ and $\{(s_\beta s_\alpha)^k\}_k$ lie in distinct equivalence classes. In fact, considering another action of (W, S) defined by another bi-linear form B' such that $B'(\alpha, \beta) < -1$, then the limit set $\Lambda_{B'}(W') \subset \Lambda_{B'}(W)$ consists of two points. In this case the limit points of $\{(s_\alpha s_\beta)^k \cdot o\}_k$ and $\{(s_\beta s_\alpha)^k \cdot o\}_k$ are distinct. On the other hand the map $\partial_G(W, S) \longrightarrow \Lambda_{B'}(W)$ is well defined hence F' cannot be an injection.

The cases (ii) and (iii).

It is known that a tangent point $p \in \mathrm{conv}(\widehat{\Delta}') \cap \partial D$ in the Case (ii) for some $\Delta' \subset \Delta$ can be expressed as the intersection of $\{H_\alpha \mid \alpha \in \Delta'\}$. We define a set PF of such points:

$$PF = \left\{p \in \partial D \;\middle|\; \exists \Delta' \subset \Delta \text{ s.t. } \{p\} = (\cap_{\alpha \in \Delta'} H_\alpha) \cap \left(\cap_{\delta \in \Delta \setminus \Delta'} P_\delta\right) \right\}.$$

Here H_α denotes a hyperplane $\{v \in V_1 \mid B(v, \alpha) = 0\}$. Then we notice that PF is the set of vertices of K' which are on ∂D by Proposition 4.6 (b).

Definition 6.2. Let (X, d) be a CAT(0) space. Fix a point $o \in X$ and take $k \in \mathbb{R}$. For $\xi \in \partial X$, we take a geodesic c from x to ξ. A *horoball* at ξ with k (based at o) is a set

$$O_{\xi,k} = \left\{x \in X \;\middle|\; \lim_{t \longrightarrow \infty} d(c(t), x) - t < k\right\}.$$

The boundary of a horoball $\partial O_{\xi,k}$ is called a *horosphere*, that is,

$$\partial O_{\xi,k} = \left\{x \in X \;\middle|\; \lim_{t \longrightarrow \infty} d(c(t), x) - t = k\right\}.$$

The function $b_c(x) := \lim_{t \longrightarrow \infty} d(c(t), x) - t$ defining the horoball is said to be a *Busemann function* associated with c. It is known that Busemann functions are well defined, convex and 1-Lipschitz. We remark that $O_{\xi,k} \subset O_{\xi,k'}$ for $k < k'$ and $O_{p,k}$ tends to p for $k \longrightarrow -\infty$. In this paper, we always take the normalized eigenvector for the negative eigenvalue of B as the base point o.

Lemma 6.3. *There exists $k \in \mathbb{R}$ such that for any $p, p' \in PF$ and $w \in W$, if $O_{p,k} \neq w \cdot O_{p',k}$ then*

$$O_{p,k} \cap w \cdot O_{p',k} = \emptyset.$$

Fix a constant k which is smaller than the constant in the claim of Lemma 6.3. Let $o \in D$ be the eigenvector corresponding to the negative eigenvalue of B as a basepoint. Then $o \in K'$ by [13, Lemma 5]. For each $p \in PF$, we take a horoball at p with k (based at o) and denote it by O_p. By Proposition 4.6 we have an affine special subgroup corresponding to each $p \in PF$ uniquely. If $W' \subset W$ is an affine subgroup corresponding to $p \in PF$ then $w \cdot O_p = O_{w \cdot p} = O_p$ for any $w \in W'$ since p is fixed by W'. We set $O := \{O_p\}_{p \in PF}$.

We remove the orbits of O from D and denote it by D'':

$$D'' = D' \setminus W \cdot O.$$

Note that D'' is closed in D because O and $R = D \setminus \operatorname{conv}(\widehat{\Delta})$ are open. The following is obvious.

Lemma 6.4. *The set D'' is invariant under the normalized action of W.*

We define $K'' := K \cap D''$. Then we can assume that $o \in K''$ by taking sufficiently small k. Recall that O contains all horoballs at the vertices of $\overline{K}$ which lie on ∂D. This indicates that $\overline{K''}$ is bounded closed set hence compact since D is proper. Since K is a fundamental region of the normalized action, Lemma 6.4 says that K'' is a fundamental region of the normalized action on D''. Define a metric d' on D'' by letting $d'(x, y)$ be the minimum length of a path in D'' connecting x and y. Now we assume that k is small enough so that the geodesic arc between o and $s \cdot o$ is in D'' for each $s \in S$.

Proposition 6.5. *W acts on (D'', d') geometrically.*

We need the hyperbolic geometry to see how the metric d' differs from the metric d. By diagonalizing B we can show that (D, d) is isometric to the hyperbolic space $(\mathbb{H}^n, d_{\mathbb{H}})$ of the upper half plane model. In $(\mathbb{H}^n, d_{\mathbb{H}})$ we can compare the hyperbolic distance of two points on a horosphere and the length of a path on that horosphere. For x, y on horosphere in $(\mathbb{H}^n, d_{\mathbb{H}})$ we denote c as an arc on horosphere joining x and y. Then we have

$$\text{the hyperbolic length of } c \leq \exp\left(\frac{d_{\mathbb{H}}(x, y)}{2}\right),$$

and hence

$$2\,(\log d'(x, y)) \leq d(x, y). \tag{1}$$

Lemma 6.6. *For a Coxeter group W of type $(n-1, 1)$, there exists a constant $C > 0$ so that*

$$2(\log \ell(w)) - C \leq d(o, w \cdot o)$$

for all $w \in W$.

Let $F : W \longrightarrow D''$ be the quasi isometry defined by $F(w) = w \cdot o$ for every $w \in W$ and if $w = w's$ for some $s \in S$ then F maps the edge joining the vertices $w, w' \in W$ to the geodesic $[w \cdot o, w' \cdot o]$.

We remind the following fact. Let (X, d) be a δ-hyperbolic space. For any $x, y, o \in X$, let z be an arbitrary point on a geodesic connecting x, y. In a δ-hyperbolic space, by the definition, $\delta \geq \min\{d(z, [o, x]), d(z, [o, y])\}$. Hence we have $d(o, z) \geq (x|y)_o$. If z is the nearest point of a geodesic $[x, y]$ from o, then we obtain $(x|y)_o \geq d(o, z) - \delta$. Thus

$$d(o, z) \geq (x|y)_o \geq d(o, z) - \delta$$

for such a point. This estimate is the key to prove the following.

Proposition 6.7. *Assume that W includes rank $m > 2$ cusps. Let $F : W \longrightarrow D''$ be the quasi isometry defined by $F(w) = w \cdot o$ for every $w \in W$. Then F extends to $\widetilde{F} : \partial_G(W, S) \longrightarrow \Lambda(W)$ continuously. Moreover $\widetilde{F}$ is surjective and W-equivariant.*

This ensures the existence of the Cannon-Thurston maps for the case (ii) and (iii). □

Corollary 1.2 follows immediately from the fact that any geodesic of a special subgroup of a Coxeter group is also a geodesic of the whole group.

References

[1] O. BAKER AND T. R. RILEY, *Cannon-Thurston maps do not always exist*, Forum of Mathematics, Sigma, Volume 1, 2013, e3.

[2] M. R. BRIDSON AND A. HAEFLIGER, *Metric Spaces of Non-Positive Curvature.* Grundlehren der math. Wiss. 319, Springer-Verlag, Berlin, 1999.

[3] S. M. BUCKLEY AND S. L. KOKKENDORFF, *Comparing the Floyd and ideal boundaries of a metric space*, Trans. Amer. Math. Soc. **361** (2009), no. 2, 715–734.

[4] J. W. CANNON AND W. P. THURSTON, *Group invariant Peano curves*, Geom. Topol., 11:1315:-1355, 2007.

[5] M. DYER, C. HOHLWEG, AND V. RIPOLL, *Imaginary cones and limit roots of infinite Coxeter groups* Preprint arXiv:1303.6710.

[6] D. EGLOFF, *Uniform Finsler Hadamard manifolds*, Ann. Inst. H. Poincaré Phys. Théor. **66** (1997), no. 3, 323-357.

[7] M. GROMOV, *Metric Structures for Riemannian and Non-Riemannian Spaces*, English ed. Modern Birkhauser Classics, Birkhauser Boston Inc., Boston, MA, 2007.

[8] W. HAEMERS, *Interlacing eigenvalues and graphs*, Linear Algebra Appl. 226/228 (1995) 593-616.

[9] A. HIGASHITANI, R. MINEYAMA, AND N. NAKASHIMA, A metric analysis of infinite Coxeter groups : the case of type $(n-1,1)$ Coxeter matrices, preprint arXiv:1212.6617.

[10] C. HOHLWEG, J-P. LABBÉ, AND V. RIPOLL, *Asymptotical behaviour of roots of infinite Coxeter groups*, Preprint arXiv:1112.5415.

[11] C. HOHLWEG, J-P. PRÉAUX, V. RIPOLL, *On the Limit Set of Root Systems of Coxeter Groups and Kleinian Groups*, preprint arXiv:1305.0052.

[12] Y. MATSUDA AND S. OGUNI, *On Cannon-Thurston maps for relatively hyperbolic groups*, Preprint arXiv:1206.5868,2012

[13] R. MINEYAMA, *Cannon-Thurston maps for Coxeter groups with signature* $(n-1,1)$, preprint arXiv:1312.3174.

[14] R. MINEYAMA, *Cannon-Thurston maps for Coxeter groups with affine reflection subgroups*, preprint arXiv:1312.5017 .

[15] M. MITRA, *Cannon-Thurston Maps for Hyperbolic Group Extensions*, Topology **37**, 527-538, 1998.

[16] M. MITRA, *Cannon-Thurston Maps for Trees of Hyperbolic Metric Spaces*, J.Differential Geom. 48, pages 135-164, 1998.